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ON AUGMENTED SCHOTTKY SPACES AND INTERCHANGE OPERATORS

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§0. Introduction.

Problem 1. Let S be a compact Riemann surface with nodes. Does there exist a point in an augmented Schottky space representing the surface S ?

Problem 2. We give a point τ in an augmented Schottky space $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ associated with a basic system of Jordan curves $\tilde{\Sigma}_0$, which represents a compact Riemann surface S with nodes. Then for any sequence of points $\{\tau_n\}$ in the Schottky space $\mathfrak{S}_g(\tilde{\Sigma}_0)$ tending to the point τ, does the Riemann surface $S(\tau_n)$ represented by τ_n converge to S as marked surfaces as $n \to \infty$?

The answer to Problem 1 is affirmative:

THEOREM 1. There exists a point in an augmented Schottky space which represents a given Riemann surface with nodes.

The answer to Problem 2 is negative in the general case, namely in the case where $\tilde{\Sigma}_0$ is a basic system of Jordan curves. However the answer is affirmative in a special case, namely in the case where $\tilde{\Sigma}_0$ is a standard system of Jordan curves. Now the following question arises: To what Riemann surface does the sequence of Riemann surfaces $\{S(\tau_n)\}$ converge as marked surface as $n \to \infty$ in the general case ?

THEOREM 2. Given a point $\tau \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$. Then there exists a sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ tending to τ such that $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces.

THEOREM 3. Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed basic system of Jordan curves for $\langle G_0 \rangle$, respectively. Given a point $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Let $\tilde{\Sigma}_0^*$, I^*, and J^* be a basic system of loops, a subset of I, and a subset of J, respectively, obtained from $\tilde{\Sigma}_0$, I and J by applying certain interchange operators. Let $\tau^* \in \delta^{I^*,J^*}\mathfrak{S}_g(\tilde{\Sigma}_0^*)$ be a point representing a compact Riemann surface with $|I^*| + |J^*|$ nodes. Then there exists the following sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$:

$$\tau_n \to \tau \quad \text{and} \quad S(\tau_n) \to S(\tau^*) \quad \text{as} \quad n \to \infty ,$$

as marked surfaces.

§ 1. Definitions.

DEFINITION 1. Let C_1, C_{g+1}; C_2, C_{g+2}; $\cdots\cdots$; C_g, C_{2g} be a set of $2g$ mutually disjoint Jordan curves on the Riemann sphere $\hat{\mathbb{C}}$ which comprize the boundary of a $2g$-ply connected region ω. Suppose there are g Möbius transformations $A_1, \cdots\cdots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \emptyset$ $(1 \leq j \leq g)$. Then A_j $(j=1,2.\cdots,g)$ generates a marked Schottky group $\langle G\rangle = \langle A_1, A_2, \cdots, A_g\rangle$. $C_1, \cdots, C_{2g}$ are called defining curves of $\langle G\rangle$.

We say two marked Schottky groups $\langle G\rangle = \langle A_1, \cdots, A_g\rangle$ and $\langle\hat{G}\rangle = \langle\hat{A}_1, \cdots, \hat{A}_g\rangle$ being equivalent if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ $(j=1,2,\cdots,g)$, and we denote it by $\langle G\rangle \sim \langle\hat{G}\rangle$.

DEFINITION 2. The Schottky space of genus g, denoted by $\mathfrak{S}_g$, is the set of all equivalent classes of Schottky groups of genus $g \geqq 1$.

DEFINITION 3. Let $C_1, \cdots, C_{2g}$ be defining curves of $\langle G\rangle = \langle A_1, \cdots, A_g\rangle$. If mutually disjoint Jordan curves $C_1, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3}$ on $\hat{\mathbb{C}}$ have the following properties (i) and (ii), then we call $\tilde{\Sigma} = \{C_1, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3}\}$ a basic system of Jordan curves (B.S.J.C.) for $\langle G\rangle$: (i) C_{2g+j} $(j=1,\cdots,2g-3)$ lie in ω. (ii) Each component of $\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{2g-3} C_{2g+j}$ is a triply connected domain. In particular, if a B.S.J.C. $\tilde{\Sigma}$ has the following property (iii), we call $\tilde{\Sigma}$ a standard system of Jordan curves (S.S.J.C.) for $\langle G\rangle$: (iii) For each $i=1,2,\cdots,g$ and $j=1,2,\cdots,2g-3$, C_i and C_{g+i} lie on the same side of C_{2g+j}. See Examples 1 and 2 on p.13.

DEFINITION 4. Let S be a compact Riemann surface. We call the set $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ of loops on S having the following property a basic system of loops (B.S.L.) : Each component of $S \setminus \bigcup_{i=1}^{g} \alpha_i \setminus \bigcup_{j=1}^{2g-3} \gamma_j$ is a planar and triply connected region. If, in particular, the number of nondividing loops is equal to g, we call a B.S.L. Σ a <u>standard system of loops</u> (S.S.L.).

Let $\Omega(G)$ be the region of discontinuity of $\langle G\rangle$. Let Π: $\Omega(G) \to \Omega(G)/\langle G\rangle = S$ be the natural projection. If $\tilde{\Sigma} = \{C_1,\cdots,C_{2g};C_{2g+1},\cdots,C_{4g-3}\}$ is a B.S.J.C. (resp. S.S.J.C.), then the projection $\Sigma = \Pi(\tilde{\Sigma}) = \{\alpha_1,\cdots,\alpha_g;\gamma_1,\cdots,\gamma_{2g-3}\}$, $\alpha_i = \Pi(C_i)$ and $\gamma_j = \Pi(C_{2g+j})$, is a B.S.L. (resp. S.S.L.). We call Σ the <u>projection</u> of $\tilde{\Sigma}$. See Examples 1 and 2 on p.13.

§2. <u>Introduction of new coordinates to $\mathfrak{S}_g$</u>.

We fix a marked Schottky group $\langle G_0\rangle = \langle A_{1,0},\cdots,A_{g,0}\rangle$. Let $\tilde{\Sigma}_0 = \{C_{1,0},\cdots,C_{2g,0};C_{2g+1,0},\cdots,C_{4g-3,0}\}$ be a fixed B.S.J.C. for $\langle G_0\rangle$. Let $\langle G\rangle = \langle A_1,\cdots,A_g\rangle$ be a marked Schottky group. Let λ_j ($|\lambda_j| > 1$), p_j and p_{g+j} be the multiplier, the repelling and the attracting fixed points of A_j, respectively. We normalize $\langle G\rangle$ by setting $p_1 = 0$, $p_{g+1} = \infty$ and $p_2 = 1$. Then a point in the Schottky space $\mathfrak{S}_g$ is identified with

$$\tilde{\tau} = (\lambda_1,\cdots,\lambda_g,p_{g+2},p_3,p_{g+3},\cdots,p_g,p_{2g}) \in \mathbf{C}^{3g-3}.$$

Now we will introduce new coordinates with respect to $\tilde{\Sigma}_0$:

$$\tau = (t_1,t_2,\cdots,t_g,\rho_1,\cdots,\rho_{2g-3}) \in \mathbf{C}^{3g-3}.$$

First define t_i by setting $t_i = 1/\lambda_i$ $(i=1,\cdots,g)$. Thus $t_i \in D^* = \{z \mid 0 < |z| < 1\}$. Next in order to define ρ_j associated with $C_{2g+j} = C(i_0,i_1,\cdots,i_\mu) \in \tilde{\Sigma}_0$ $(j=1,2,\cdots,2g-3)$, we determine integers $k(j)$, $\ell(j)$, $m(j)$ and $n(j)$ which are ≥ 1 and $\leq 2g$ as follows, where $C(i_0,i_1,\cdots,i_\mu)$ is the multi-suffix of C_{2g+j} (see [4] for the definition): $k(j) = 1$, $C_{\ell(j)} = C(i_0,i_1,\cdots,i_{\mu-1},1-i_\mu,0,\cdots,0)$, $C_{m(j)} = C(i_0,i_1,\cdots,i_\mu,0,\cdots,0)$ and $C_{n(j)} = C(i_0,i_1,\cdots,i_\mu,0,\cdots,0)$. The coordinate ρ_j is now defined as follows:

$$(p_{k(j)},p_{\ell(j)},p_{m(j)},p_{n(j)}) = (0,1,\infty,\rho_j)\ ,$$

where (a,b,c,d) means the cross ratio of $a,b,c,$ and d.

We define a mapping ϕ by $\phi(\langle G\rangle) = \tau$. We note that if $\langle G\rangle \sim \langle\hat{G}\rangle$, then $\phi(\langle G\rangle) = \phi(\langle\hat{G}\rangle)$. We denote by $\mathfrak{S}_g(\tilde{\Sigma}_0)$ the set

$$\mathfrak{S}_g(\tilde{\Sigma}_0) = \{\tau = \phi(\langle G\rangle) \mid \langle G\rangle \in \mathfrak{S}_g\}.$$

Then $\mathfrak{S}_g(\tilde{\Sigma}_0) \cong \mathfrak{S}_g$ and $\mathfrak{S}_g(\tilde{\Sigma}_0) \subset D^{*g} \times (\mathbf{C}\setminus\{0,1\})^{2g-3}$. We call $\mathfrak{S}_g(\tilde{\Sigma}_0)$ the <u>Schottky space associated with</u> $\tilde{\Sigma}_0$.

§3. <u>Augmented Schottky spaces</u>.

Let $\langle G_0\rangle$ and $\tilde{\Sigma}_0$ be a fixed Schottky group and a fixed B.S.J.C. as in §2.

<u>DEFINITION 5</u>. We say $C_{2g+j} = C(i_1,\cdots,i_\mu)$ (resp. $C_i = C(j_1,\cdots,j_\sigma)$) is <u>behind</u> $C_{2j+\ell} = C(i'_1,\cdots,i'_\nu)$ if $\nu < \mu$ and $i_k = i'_k$ $(k=1,2,\cdots,\nu)$ (resp. $\nu < \sigma$ and $j_k = i'_k$ $(k=1,2,\cdots,\nu)$), and denote the fact $C_{2g+\ell} < C_{2g+j}$ (resp. $C_{2g+\ell} < C_i$). Otherwise, we say that C_{2g+j} (resp. C_i) is <u>not behind</u> $C_{2g+\ell}$ and we denote the fact by $C_{2g+\ell} \not< C_{2g+j}$ (resp. $C_{2g+\ell} \not< C_i$).

We define the ordered cycle corresponding to α_i as follows.

We denote the shortest path from C_i to C_{g+i} on the tree of $\tilde{\Sigma}_0$ by

$$(1) \quad C_i,\ C^{\delta(1)}_{2g+i(1)},\ C^{\delta(2)}_{2g+i(2)}, \cdots\cdots,\ C^{\delta(k)}_{2g+i(k)},\ C_{g+i}$$

(see [4] and Fig. 1 on p.13 in this paper for trees.) Here $\delta(\ell)$ $(\ell=1,2,\cdots,k)$ are determined by $\delta(\ell) = +1$ or $\delta(\ell) = -1$ according as $C_{2g+\ell} < C_{g+i}$ or $C_{2g+\ell} < C_i$.

<u>DEFINITION 6</u>. The projection

$$(\alpha_i\ ;\ \gamma^{\delta(1)}_{i(1)}, \cdots\cdots, \gamma^{\delta(k)}_{i(k)})$$

of (1) onto $S_0 = \Omega(G_0)/<G_0>$ is called the <u>ordered cycle</u> corresponding to α_i, and is denoted by $L_{0,i}$.

Let I be a subset of $\{1,2,\cdots,g\}$ and J a subset of $\{1, 2,\cdots,2g-3\}$. We denote by $|I|$ and $|J|$ the cardinarity of I and J, respectively. Let $L_{0,j(1)}, L_{0,j(2)},\cdots\cdots,L_{0,j(t)}$ be the complete list of cycles containing γ^{δ}_j, and let $\alpha_{0,k}$ be the "α-loops" contained in $L_{0,k}$ $(1 \leqq k \leqq t)$, where $t = t(j)$ depends on j. We define the subset $I(J)$ of $\{1,2,\cdots,g\}$ by

$$I(J) = \{i \in \{1,2,\cdots,g\} \mid \alpha_{0,i} \text{ is contained in } L_{0,j(k)} \text{ for some } k\ (1 \leqq k \leqq t(j)) \text{ and for some } j \in J\}.$$

<u>Remark</u>. If $\tilde{\Sigma}_0$ is a S.S.J.C. , then $I(J) = \emptyset$.

We define the following sets $X = \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ with $I \supset I(J)$:

(i) When $I = J = \emptyset$, we define X as $\mathfrak{S}_g(\tilde{\Sigma}_0)$, the Schottky space associated with $\tilde{\Sigma}_0$.

(ii) When $I \neq \emptyset$, $j = \emptyset$, we define X as follows:

$\delta^{I,\emptyset}\mathfrak{S}_g(\tilde{\Sigma}_0)$ = $\{\tau=(t_1,\cdots,t_g,\rho_1,\cdots,\rho_{2g-3}) \mid t_i=0\ (i\in I),\ t_i\neq 0\ (i\notin I),\ \rho_j\neq 1\ (j=1,\cdots,2g-3)$, and τ represents a Rienann surface with nodes such that only α_i $(i\in I)$ are nodes$\}$.

(iii) When $I = \emptyset$, $J \neq \emptyset$, we define X as follows:

$\delta^{\emptyset,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ = $\{\tau=(t_1,\cdots,t_g,\rho_1,\cdots,\rho_{2g-3}) \mid t_i\neq 0\ (i=1,\cdots,g),\ \rho_j=1\ (j\in J),\ \rho_j\neq 1\ (j\notin J)$ and τ represents a Riemann surface with nodessuch that only γ_j are nodes$\}$.

(iv) When $I\supset I(J) \neq \emptyset$, X is defined as follows:

$\delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ = $\{\tau=(t_1,\cdots t_g,\rho_1,\cdots,\rho_{2g-3}) \mid t_i=0\ (i\in I),\ t_i\neq 0\ (i\notin I),\ \rho_j=1\ (j\in J),\ \rho_j\neq 1\ (j\notin J)$ and ρ represents a compact Riemann surface such that only α_i $(i\in I)$ and γ_j $(j\in J)$ are nodes$\}$.

<u>DEFINITION 7</u>.

$$\hat{\mathfrak{S}}^*_g(\tilde{\Sigma}_0) = \bigcup \{\delta^{I,J}\mathfrak{S}_g(\Sigma_0) \mid I\subset\{1,2,\cdots,g\},\ J\subset\{1,2,\cdots,3g-3\} \text{ with } I\supset I(J)\}$$

is called the <u>augmented Schottky space associated with</u> $\tilde{\Sigma}_0$.

<u>Remark</u>. Let $S(\tau)$ be the Riemann surface represented by τ. $\{S(\tau) \mid \tau\in\hat{\mathfrak{S}}^*_3(\tilde{\Sigma}_0)\}$ is the sets of all Riemann surfaces in Fig.2 and Fig.3 in the cases of Example 1 and Example 2, respectively.

§ 4. Interchange operators.

For simplicity, we will only consider interchange operators in the case of Example 1 (see Fig.4). For detail, see Sato [5]. Choose j with $I(\{j\}) \neq \emptyset$. Let $i \in I(\{j\})$. For these i and j, we introduce the interchange operators $I_g(i,j)$.

Remark. Since $I(J)$ is always empty in the case where $\tilde{\Sigma}$ is a S.S.J.C., we can not define an interchange operator in this case.

For simplicity, we only consider $I_g(1,2)$, which is defined as follows (see Fig.4 on p.15): For a B.S.J.C. $\tilde{\Sigma}$,

$$I_g(1,2)(\tilde{\Sigma}) = \tilde{\Sigma}^* = \{C_1^*, C_2^*, \cdots, C_6^*; C_7^*, C_8^*, C_9^*\},$$

where $C_1^* = A_1^{-1}(C_8)$, $C_2^* = A_1^{-1}(C_2)$, $C_3^* = C_3$, $C_4^* = C_8$, $C_5^* = C_5$, $C_6^* = C_6$, $C_7^* = C_7$, $C_8^* = C_1$, and $C_9^* = C_9$.

For a B.S.L. $\Sigma = \{\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2, \gamma_3\}$, $I_g(1,2)(\Sigma) = \{\alpha_1^*, \alpha_2^*, \alpha_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^*\}$, where $\alpha_1^* = \gamma_2$, $\alpha_2^* = \alpha_2$, $\alpha_3^* = \alpha_3$, $\gamma_1^* = \gamma_1$, γ_2^* α_1, $\gamma_3^* = \gamma_3$.

For ordered cycles L_1, L_2 and L_3, $L_1^* = I_g(1,2)(L_1) = (\alpha_1^*; \gamma_2^*, \gamma_1^*)$, $L_2^* = I_g(1,2)(L_2) = (\alpha_2^*; \gamma_2^*, \gamma_1^*, \gamma_3^*)$ and $L_3^* = I_g(1,2)(L_3) = (\alpha_3^* : \gamma_3^{*-1}, \gamma_1^{*-1})$, where we write γ_j^* for γ_j^{*+1} for simplicity.

For a marked Schottky group $\langle G\rangle = \langle A_1, A_2, A_3\rangle$, $\langle G^*\rangle = I_g(1,2)(\langle G\rangle) = \langle A_1^*, A_2^*, A_3^*\rangle$, where $A_1^* = A_1$, $A_2^* = A_2A_1$, $A_3^* = A_3$.

We obtain Theorem 1 by using interchange operators. See Sato [5] for details.

§5. Relations between limits of Schottky groups and limits of Riemann surfaces.

Here we will consider Problem 2. Let S be a compact Riemann surface of genus g with or without nodes. We denote by $N(S)$ the set of all nodes on S. We assume that each component of $S \setminus N(S)$ has the Poincaré metric. The Poincaré metric $\lambda(z)|dz|$ on S is defined as the Poincaré metric on each component of $S \setminus N(S)$.

DEFINITION 8. If the following conditions are satisfied, a sequence of Riemann surfaces $\{S_n\}$ converges to a surface S as marked surfaces: There exists a locally quasiconformal mapping $\phi_n : S \setminus N(S) \to S_n \setminus P(S_n)$ such that (i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converges to $\phi(z)|dz|$ on every compact subset of $S \setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on S_n and S, respectively, (ii) ϕ_n maps a deleted neighborhood $N(\alpha_i)\setminus\{\alpha_i\}$ (resp. $N(\gamma_j)\setminus\{\gamma_j\}$) of α_i (resp. γ_j) to a deleted neighborhood $N(\alpha_{i,n})\setminus\{\alpha_{i,n}\}$ (resp. $N(\gamma_{j,n})\setminus\{\gamma_{j,n}\}$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and (iii) ϕ_n maps a neighborhood $N(\alpha_i)$ (resp. $N(\gamma_j)$) of α_i (resp. γ_j) to a neighborhhod $N(\alpha_{i,n})$ (resp. $N(\gamma_{j,n})$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \notin N(S)$ (resp. $\gamma_j \notin N(S)$), where $P(S_n) = f_n^{-1}(N(S))$ and $f_n: S_n \to S$ is a deformation.

By constructing locally quasiconformal mappings, we have Theorem 2. See Sato [6] for details.

Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed B.S.J.C. for $\langle G_0 \rangle$, respectively. Set $S_0 = \Omega(G_0)/\langle G_0 \rangle$. Given a point $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then $S(\tau)$ is a compact Riemann surface with $|I| + |J|$ nodes of genus g. We define the following sets: $J_1 = \{j \in J \mid \gamma_j$ is a dividing loop on $S_0\}$, $J_2 =$ any subset of $J \setminus J_1$, $\tilde{\Sigma}_1 = I_g(i_{k(1)}, j_{\ell(1)})(\tilde{\Sigma}_0)$ with $i_{k(1)} \in I(\{j_{\ell(1)}\})$, $j_{\ell(1)} \in J_2$ and $J_{21} = J_2 \setminus \{j_{\ell(1)}\}$. Choose $j_{\ell(2)} \in J_{21}$ such that $I_1(\{j_{\ell(2)}\}) \cap (I(J_2) \setminus \{i_{k(1)}\}) \neq \emptyset$. Set $\tilde{\Sigma}_2 = I_g(i_{k(2)}, j_{\ell(2)})(\tilde{\Sigma}_1)$ with $i_{k(2)} \in I_1(\{j_{\ell(2)}\})$, $i_{k(2)} \neq i_{k(1)}$. We set $J_{22} = J_{21} \setminus \{j_{\ell(2)}\} = J_2 \setminus \{j_{\ell(1)}, j_{\ell(2)}\}$. By the same way, we determined the following: $j_{\ell(3)}, i_{k(3)}, J_{23}, \tilde{\Sigma}_3, I_3(J_{23}); \cdots \cdots; j_{\ell(s)}, i_{k(s)}, J_{2,s}, \tilde{\Sigma}_s$: Here s is the integer satisfying the following (i) and (ii) : (i) $I_{s-1}(\{j_{\ell(s)}\}) \cap I(J_2) \setminus \{i_{k(1)}, i_{k(2)}, \cdots, i_{k(s-1)}\} \neq \emptyset$, (ii) $I_s(\{j\}) \subsetneqq \{i_{k(1)}, \cdots, i_{k(s)}\}$ for any $j \in J_2 \setminus \{j_{\ell(1)}, j_{\ell(2)}, \cdots, j_{\ell(s)}\}$.

We set $J_3 = J \setminus (J_1 \cup J_2)$, $J_4 = \{j_{\ell(1)}, j_{\ell(2)}, \cdots, j_{\ell(s)}\}$, $J_5 = J_2 \setminus J_4$, $I_1 = I \setminus I(J)$, $I_4 = \{i_{k(1)}, i_{k(2)}, \cdots, i_{k(s)}\}$, $I_3 = I_s(J_3)$, $I_5 = I \setminus (I_1 \cup I_3 \cup I_4)$, $I_6 =$ a subset of I_5, $I_7 = I_5 \setminus I_6$, $I^* = I \setminus I_7$ and $J^* = J \setminus J_4$. Then we have Theorem 3. See Sato [6] for the proof.

<u>COROLLARY</u>. Given $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then there exists a sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ such that (i) $\tau_n \to \tau$ as $n \to \infty$ and (ii) $S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces.

<u>Remark</u>. By a similar method to the proof of Theorem 2, we

have the following. If $\widetilde{\Sigma}_0$ is a S.S.J.C., then $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces for any point $\tau \in \widehat{\mathfrak{S}}_g^*(\widetilde{\Sigma}_0)$ and for any sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\widetilde{\Sigma}_0)$ with $\tau_n \to \tau$.

§6. Appendices.

We will consider the following in the forthcoming papers [7,8].

1. Properties of interchange operators. There are five kind of interchange operators as follows: (1) $I_g(\alpha_i, \alpha_i^{-1}) = I_g(C_i, C_{g+i})$, (2) $I_g(\alpha_i, \alpha_j) = I_g(C_i, C_j)$, (3) $I_g(\gamma_j, \gamma_j^{-1}) = I_g(C_{2g+j}^+, C_{2g+j}^-)$, (4) $I_g(\gamma_i, \gamma_j) = I_g(C_{2g+i}, C_{2g+j})$ and (5) $I_g(\alpha_i, \gamma_j) = I_g(C_i, C_{2g+j})$. Here we only considered and used interchanged operators in case (5).

2. Relations between Nielsen isomorphisms and interchange operators. Here Nielsen isomorphims are

$$N_1(A_1, A_i) : \langle A_1, A_2, \cdots, A_i, \cdots, A_g\rangle \to \langle A_i, A_2, \cdots, A_1, \cdots, A_g\rangle .$$

$$N_2(A_1, A_1^{-1}) : \langle A_1, A_2, \cdots, A_g\rangle \to \langle A_1^{-1}, A_2, \cdots, A_g\rangle .$$

$$N_3(A_1, A_2) : \langle A_1, A_2, A_3, \cdots, A_g\rangle \to \langle A_1, A_1A_2, A_3, \cdots, A_g\rangle .$$

3. Boundary behavior of the space of marked Schottky groups of real type of genus 2. We say $\langle G\rangle = \langle A_1, A_2\rangle$ a schottky group of real type if $A_1, A_2 \in SL(2,\mathbf{R})$.

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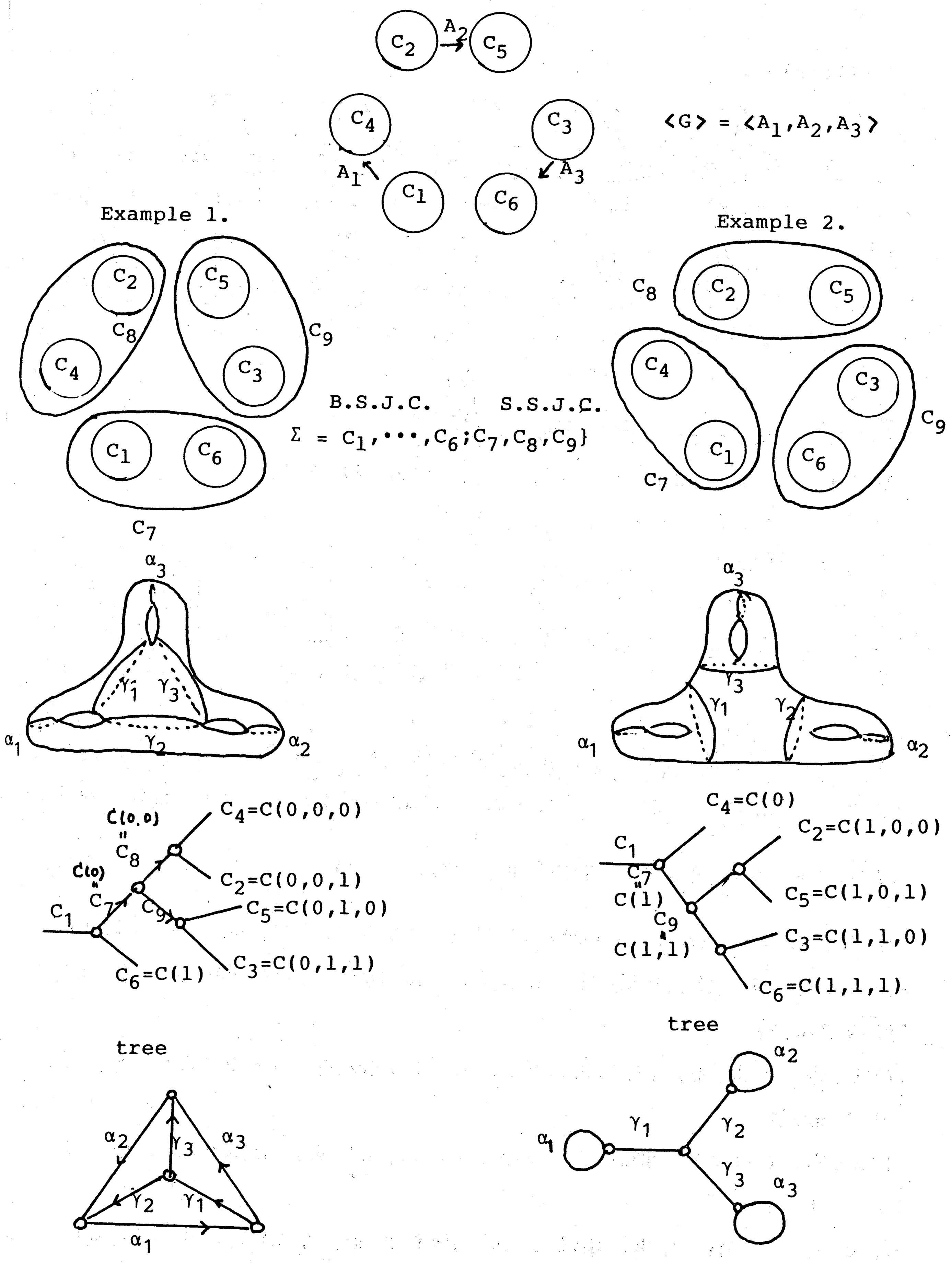


$\langle G\rangle = \langle A_1, A_2, A_3\rangle$
Example 1.
Example 2.
B.S.J.C.
S.S.J.C.
$\Sigma = C_1, \cdots, C_6; C_7, C_8, C_9\}$
$C_4 = C(0,0,0)$
$C_2 = C(0,0,1)$
$C_5 = C(0,1,0)$
$C_3 = C(0,1,1)$
$C_6 = C(1)$
$C(0,0) = C_8$
$C(0) = C_7$
$C_4 = C(0)$
$C_2 = C(1,0,0)$
$C_5 = C(1,0,1)$
$C_3 = C(1,1,0)$
$C_6 = C(1,1,1)$
$C_7 = C(1)$
$C_9 = C(1,1)$
tree
tree

Fig.1

Example 1.

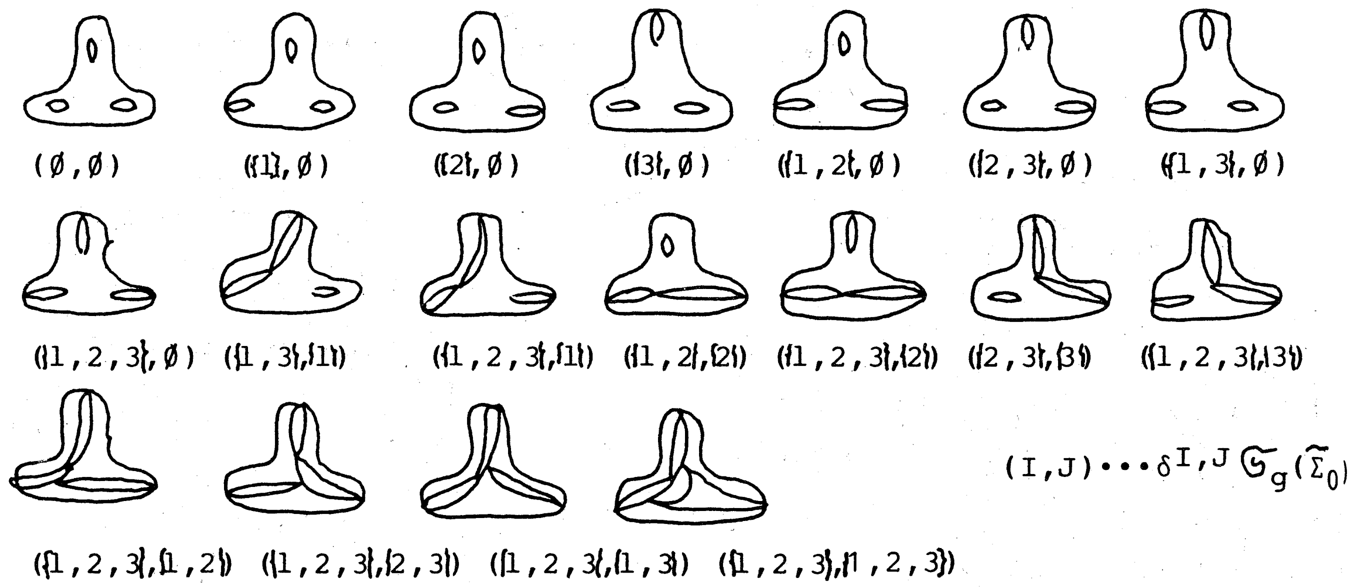
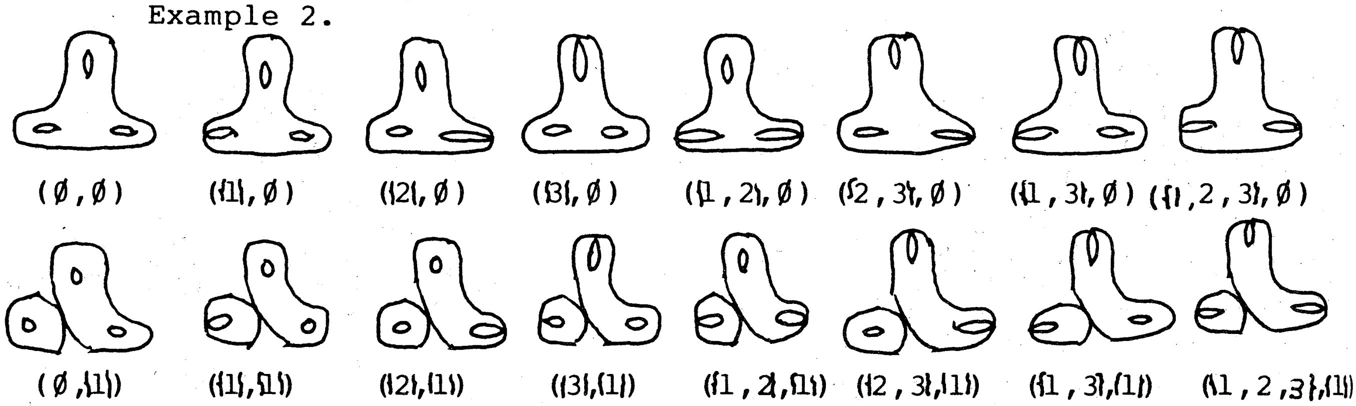


Fig.2.

Example 2.



(∅,{2}), ({1},{2}), ({2},{2}), ({3},{2}), ({1,2},{2}), ({2,3},{2}), ({1,3},{2}), ({1,2,3},{2})

(∅,{3}), ({1},{3}), ({2},{3}), ({3},{3}), ({1,2},{3}), ({2,3},{3}), ({1,3},{3}), ({1,2,3},{3})

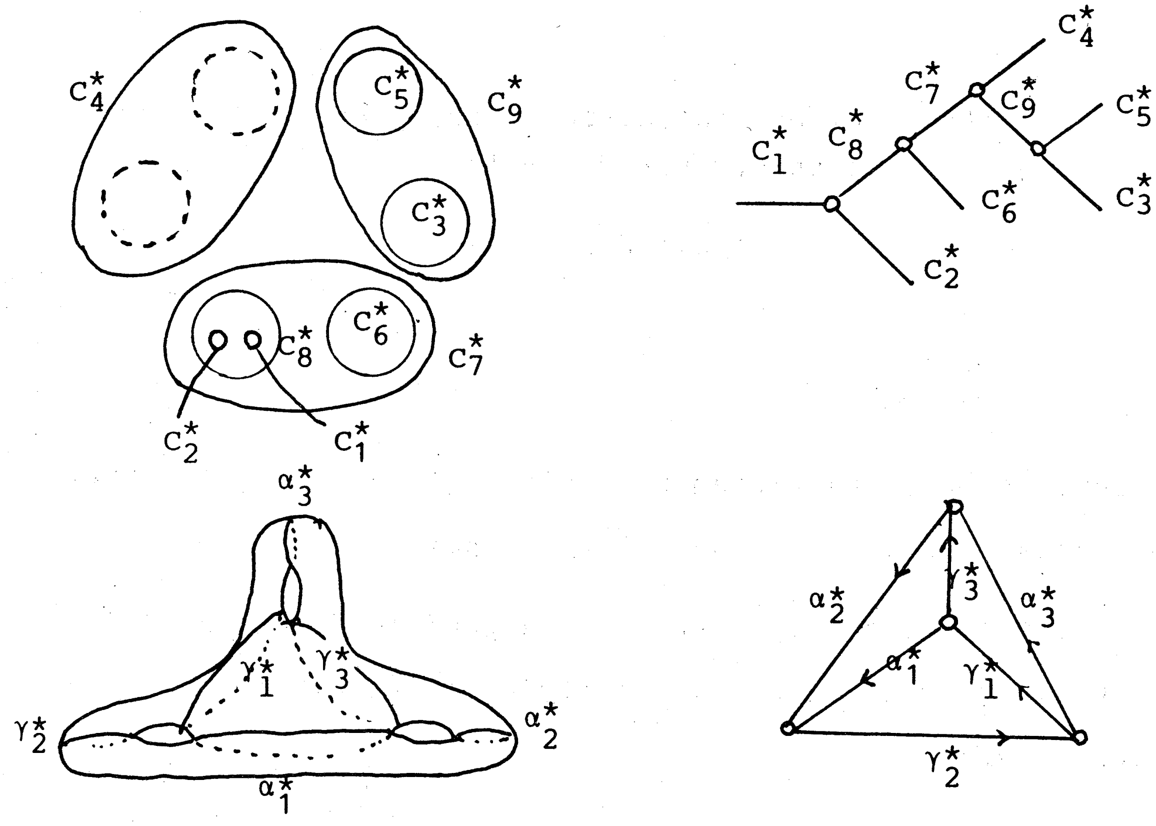
(∅,{1,2}), ({1},{1,2}), ({2},{1,2}), ({3},{1,2}), ({1,2},{1,2}), ({2,3},{1,2}), ({1,3},{1,2}),
({1,2,3},{1,2})

(∅,{2,3}), ({1},{2,3}), ({2},{2,3}), ({3},{2,3}), ({1,2},{2,3}), ({2,3},{2,3}), ({1,3},{2,3}),
({1,2,3},{2,3})

(∅,{1,3}), ({1},{1,3}), ({2},{1,3}), ({3},{1,3}), ({1,2},{1,3}), ({2,3},{1,3}), ({1,3},{1,3}),
({1,2,3},{1,3})

(∅,{1,2,3}), ({1},{1,2,3}), ({2},{1,2,3}), ({3},{1,2,3}), ({1,2},{1,2,3}), ({2,3},{1,2,3}),
({1,3},{1,2,3}), ({1,2,3},{1,2,3})

Fig.3.



$\tilde{\Sigma}^* = I_g(1,2)(\tilde{\Sigma}) = \{c_1^*, \cdots, c_6^*; c_7^*, c_8^*, c_9^*\}$

$\Sigma^* = I_g(1,2)(\Sigma) = \{\alpha_1^*, \alpha_2^*, \alpha_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^*\}$

Fig,4.